Book Review

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S. HELGASON (1984). *Groups and Geometric Analysis* (Integral Geometry, Invariant Differential Operators and Spherical Functions), Academic Press, New York, 654 pp.

Nowadays, harmonic analysis on Riemannian symmetric spaces (of Euclidean, compact or non-compact type) is a rather advanced field with many different aspects. Helgason's *Groups and Geometric Analysis* offers an introduction to those aspects which have been among the main research interests of the author in the last thirty years. The diversity of subjects treated is great. Nevertheless the author has managed to achieve coherence of presentation by clearly putting forward a few main themes and basic problems. To illustrate this I intend to systematically go through the contents of the book.

Two main themes of harmonic analysis dominate the first part of the book: firstly the theme of integral transforms (mainly Radon transforms, a few orbital integrals), and secondly that of invariant differential operators. The second part of the book deals with the analysis of spherical functions on Riemannian symmetric spaces, especially those of non-compact type: it provides a beautiful illustration of the themes mentioned.

All of the above are illuminated in an introductory chapter which gives a detailed treatment of the three basic examples: $\mathbb{R}^2 \cong M(2)/O(2)$ = group of isometries of $\mathbb{R}^2$ modulo the stabilizer of the origin (Euclidean type), $S^2 \cong O(3)/O(2)$ (compact type: spherical harmonics) and finally the hyperbolic disk $D = \{z \in \mathbb{C};\ |z| < 1\}$ viewed as the homogeneous space $SU(1,1)/SO(2)$ (of non-compact type). A reader having no background in Lie group theory will get an excellent impression of the role group actions play in harmonic analysis on these spaces.

The next chapter gives a thorough treatment of the d-Radon transform (integration over d-planes) in $\mathbb{R}^n$. Treated are: the inversion, the support and the Plancherel theorem. Applications to PDE's and radiography (X-ray transform) are briefly mentioned. After this the group theoretic structure which underlies the Radon transform is analyzed and formulated in greater generality. Since I found it illuminating, I'll briefly discuss this point of view here.

Let $X=\mathbb{R}^n$ and let Y be the smooth manifold of all hyperplanes in $\mathbb{R}^n$. Then the $(n-1)$-Radon transform R_X is the map $C_c^\infty(X)\rightarrow C^\infty(Y)$ defined by

$$R_X f(y) = \int_{\hat{y}} f(x)dm_y(x),$$

for $f\in C_c^\infty(X)$. Here $dm_y(x)$ denotes $(n-1)$-dimensional Euclidean measure on the hyperplane $\hat{y}\subset X$ corresponding to the point $y\in Y$. There is also a dual Radon transform R_Y: $C_c^\infty(Y)\rightarrow C^\infty(X)$. If $\phi\in C_c^\infty(Y)$ then $R_Y\phi(x)$ is defined by integrating ϕ over the closed submanifold $\check{x}=\{y\in Y;\ x\in y\}$ of Y. The map R_Y is the transposed of R_X. For $f\in C_c^\infty(X)$, one has the beautiful inversion formula

$$f = \Gamma(\tfrac{1}{2})\Gamma(n/2)^{-1}(-\frac{1}{4\pi}\Delta)^{\frac{1}{2}(n-1)}R_Y R_X f,$$

involving a fractional power of the Laplacian Δ. This formula goes back to RADON [14] for $n=3$ and to JOHN [11] for $n>3$. Its generalization to the d-plane transform is due to HELGASON [7]. We'll now see how group theory enters. In a natural fashion the group $M(n)$ of isometries of $\mathbb{R}^n$ acts transitively on both X and Y. Thus $X\cong M(n)/O(n)$ and $Y\cong M(n)/F$, where $F\cong\mathbb{Z}_2\times M(n-1)$ is the stabilizer of the hyperplane $x_1=0$ in $\mathbb{R}^n$. The crucial observation now is that R_X is equivariant for the natural actions of $M(n)$ on $C_c^\infty(X)$ and $C^\infty(Y)$. Thus representation theory enters the scene. Moreover, the property of equivariance suggests a generalization of the Radon transform to more homogeneous spaces $X=G/H_X$ and $Y=G/H_Y$ for the same Lie group G. Two elements $x\in X$ and $y\in Y$ are called incident if $x\cap y\neq\varnothing$ as cosets in G. A generalized Radon transform can now be defined by integrating functions on X over sets $\hat{y}=\{x\in X;\ x \text{ and } y \text{ incident}\}$. Similarly a dual transform R_Y can be defined. By the way, if $G=U(4)$, $H_X=U(1)\times U(3)$ and $H_Y=U(2)\times U(2)$, then the maps $y\mapsto\hat{y}$ and $x\mapsto\check{x}$ are Penrose correspondences, see PENROSE [13].

Using the general set up indicated above, the author discusses the analysis of Radon transforms for the non-Euclidean Riemannian symmetric spaces of rank 1.

The second chapter deals with the algebra $\mathbb{D}(G/H)$ of invariant differential operators on a homogeneous space G/H of a Lie group G. Geometric constructions such as separation of variables and taking radial parts are discussed in generality. For Riemannian symmetric spaces G/H the algebra $\mathbb{D}(G/H)$ is analyzed in great detail. From this point on the book may be considered as a continuation of Helgason's previous book [10].

Chapter 3 deals with linear group actions (in particular by finite reflection groups), and the corresponding invariant and harmonic polynomials. At the end the Kostant-Rallis theory of adjoint orbits in a symmetric space is discussed.

Chapter 4 is devoted to the study of spherical functions and spherical transforms on a Riemannian symmetric space $X=G/K$ of the non-compact type. Here G is a real semisimple Lie group with a maximal compact subgroup K. The algebra $\mathbb{D}(X)$ of invariant differential operators is commutative. Its joint eigenspaces

$$E(X,\chi) = \{f \in C^{\infty}(X);\ Df = \chi(D)f,\ D \in \mathbb{D}(X)\}$$

are parametrized by characters $\chi \in \mathbb{D}(X)\hat{}$. The spaces $E(X,\chi)$ are invariant for the left regular representation L of G on $C^{\infty}(X)$. Basic problems put forward by the author are:

(1) to describe the joint eigenspaces $E(X,\chi)$,

(2) to determine for which $\chi \in \mathbb{D}(X)\hat{}$, the restriction of L to $E(X,\chi)$ is irreducible,

(3) to decompose functions on X in terms of joint eigenfunctions (Fourier decomposition).

Historically, the third of these problems was solved first, by Harish-Chandra [5]. The set $\mathbb{D}(X)\hat{}$ can be parametrized in a natural fashion by $a_{\mathbb{C}}^{*}/W$, where a is a maximal abelian linear subspace of the Killing orthocomplement of Lie(K) in Lie(G), and where W is the finite reflection group determined by the a-roots in Lie(G). If $\lambda \in a_{\mathbb{C}}^{*}$, then the corresponding element of $\mathbb{D}(X)\hat{}$ is denoted χ_λ. The space $E_\lambda(X) = E(X,\chi_\lambda)$ contains a unique left K-invariant function ϕ_λ with $\phi_\lambda(e)=1$, the so-called elementary or zonal spherical function. Explicitly, ϕ_λ can be given as a Radon transform of a function of exponential type. Any K-invariant function $f \in C_c^{\infty}(X)$ can be decomposed as

$$f(x) = \int_{ia^*} \phi_\lambda(x)\tilde{f}(\lambda)|c(\lambda)|^{-2} d\lambda.$$

Here $d\lambda$ denotes suitably normalized Lebesgue measure on ia^*. Moreover, $\tilde{f}(\lambda) = \int_X f(x)\phi_{-\lambda}(x)dx$ is the so called spherical Fourier transform of f. Finally, $c(\lambda)$ is the famous c-function, which occurs as leading coefficient in a converging series expansion describing the asymptotics of $\phi_\lambda(x)$ as x tends to infinity in X. Originally, Harish-Chandra proved this result in [5] for a space $\mathbb{S}$ of K-invariant rapidly decreasing functions on X (the proper analogue of the Euclidean Schwartz space), subject to two conjectures being true. One of these conjectures involved an estimate for the c-function, the other density of a space of wave packets in $\mathbb{S}$. The first conjecture was solved by GINDIKIN and KARPELEVIČ [4], who expressed the c-function as a product of quotients of Γ-functions. The other was solved by HARISH-CHANDRA [6].

Using the above inversion formula for C_c^{∞}-functions, HELGASON [8] proved a Paley-Wiener theorem for the spherical Fourier transform, except for certain estimates for the coefficients in the series expansion of ϕ_λ. The missing estimates were provided by GANGOLLI [3]. ROSENBERG [15] discovered that it was possible to first prove a support result on wave packets

$$\int_{ia^*} \phi_\lambda(x) A(\lambda) |c(\lambda)|^{-2} d\lambda,$$

with A a W-invariant entire function of Paley-Wiener type on $a_{\mathbb{C}}^*$, and to use this to give a much simpler proof of the inversion formula for C_c^∞-functions. The present book is the first to give a self contained account of these short proofs of the inversion and the Paley-Wiener theorem. It may be interesting to know that after the appearance of this book an even shorter proof of the Paley-Wiener theorem has been discovered by FLENSTED-JENSEN [2]. His proof completely avoids the consideration of asymptotics of spherical functions: instead via an ingenious variation on Hermann Weyl's unitary trick a reduction to the complex and then the Euclidean case is given. A drawback of this method is that it does not give the inversion formula.

The above questions (1) and (2), taken up first by HELGASON [9] have also given rise to some beautiful developments in the subject. In the book they are only dealt with for the case of the hyperbolic disk D, in the introductory chapter. It turns out that $E_\lambda(D)$ can be characterized as the image under a generalized Poisson transformation of the space of hyperfunctions on the boundary $\partial D = \{z \in \mathbb{C} : |z| = 1\}$: the classical integral representation of harmonic functions on the disk is a special case of this. The analogue of the above description of eigenfunctions by Poisson transformations for a general Riemannian symmetric space of the non-compact type was conjectured and partially proved by HELGASON [9] and finally proved by KASHIWARA ET AL. [12]. An excellent introduction to this material can be found in SCHLICHTKRULL [16].

The book ends with a chapter on (the relatively standard) Fourier analysis on a Riemannian symmetric space of the compact type.

Each chapter of the book concludes with a set of exercises and in addition a set of historical notes which is usually very complete and helpful. In fact I noticed only one omission: in the discussion of asymptotics of zonal spherical functions a reference to the enlightening paper of CASSELMAN and MILIČIĆ [1] is missing.

The first third of the book can certainly be used as a textbook for beginning graduate students. The rest requires a greater knowledge of Lie group theory which however nowhere goes beyond the contents of the author's previous book [10]. The present book will also be an excellent source of reference for experts. No doubt it will become a new standard in the field.

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